MATH 2020A Tutorial 3 8/10/2015

Topic:
- Change of Variable Formula
- Line Integral
- Potential Function and Gradient Field

Change of Variable Formula:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(u,v)\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv,$$

where $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ is an injective differentiable transformation, and $\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ is the Jacobian matrix.

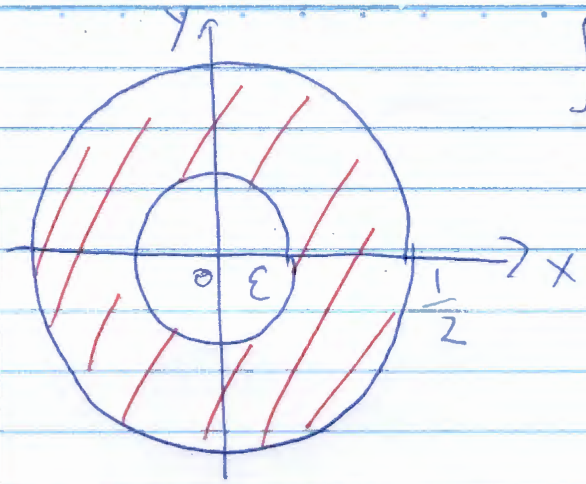
Example: 1) $\int_S g\,dA$, where $g(x,y) = \frac{1}{(x^2+y^2)\left(\log\frac{1}{\sqrt{x^2+y^2}}\right)^2}$,

$$S = \{(x,y)\in\mathbb{R}^2 \mid \varepsilon \le \sqrt{x^2+y^2} \le \tfrac{1}{2}\}$$

Ans: Use polar coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad r\in[\varepsilon,\tfrac{1}{2}], \quad \theta\in[0,2\pi]$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \quad \left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$



⇑ S in xy-plane ⇑

$$\int_S g\,dA = \int_0^{2\pi}\int_{\varepsilon}^{\frac{1}{2}} \frac{r}{r^2(\log\frac{1}{r})^2}\,dr\,d\theta$$

$$= 2\pi\int_{\varepsilon}^{\frac{1}{2}} \frac{1}{r(\log r)^2}\,dr$$

$$= 2\pi\int_{\varepsilon}^{\frac{1}{2}} \frac{1}{(\log r)^2}\,d(\log r)$$

$$= 2\pi\left[\frac{-1}{\log r}\right]_{\varepsilon}^{\frac{1}{2}}$$

$$= 2\pi\left(\frac{1}{\log 2} + \frac{1}{\log \varepsilon}\right) \quad //$$

2) $\iint_R (x^2+y^2)\,dA$, where R is the region in 1st quadrant bounded by the curves

$x^2-y^2=1,\ x^2-y^2=4,\ xy=1,\ xy=4$

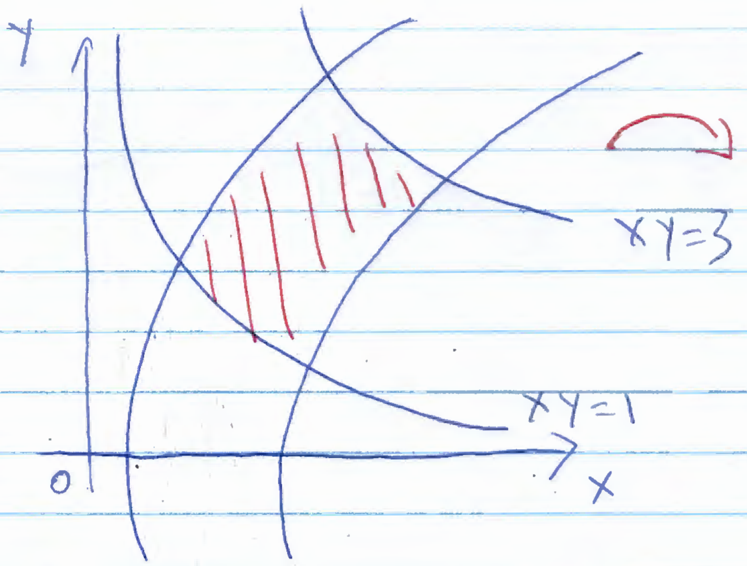
Ans: Use the transformation

$$\begin{cases} u = x^2-y^2 \\ v = xy \end{cases}, \quad \begin{matrix} u \in [1,4] \\ v \in [1,3] \end{matrix}$$

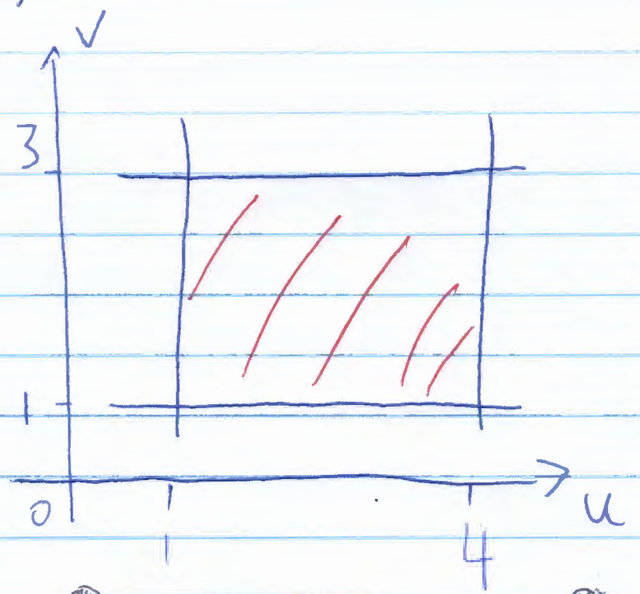
Useful Formula: $\det\dfrac{\partial(x,y)}{\partial(u,v)} = \left[\det\dfrac{\partial(u,v)}{\partial(x,y)}\right]^{-1}$

$$\det \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2)$$

$$\therefore \det \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)}$$



R in xy-plane



R in uv-plane

$$\therefore \iint_R (x^2+y^2)\,dA = \int_1^3 \int_1^4 \frac{1}{2}\,du\,dv = 3$$

3) $\iint_A e^{\frac{x-y}{x+y}}\,dx\,dy$, where A is the region bounded by x,y-axis and $x+y=1$.

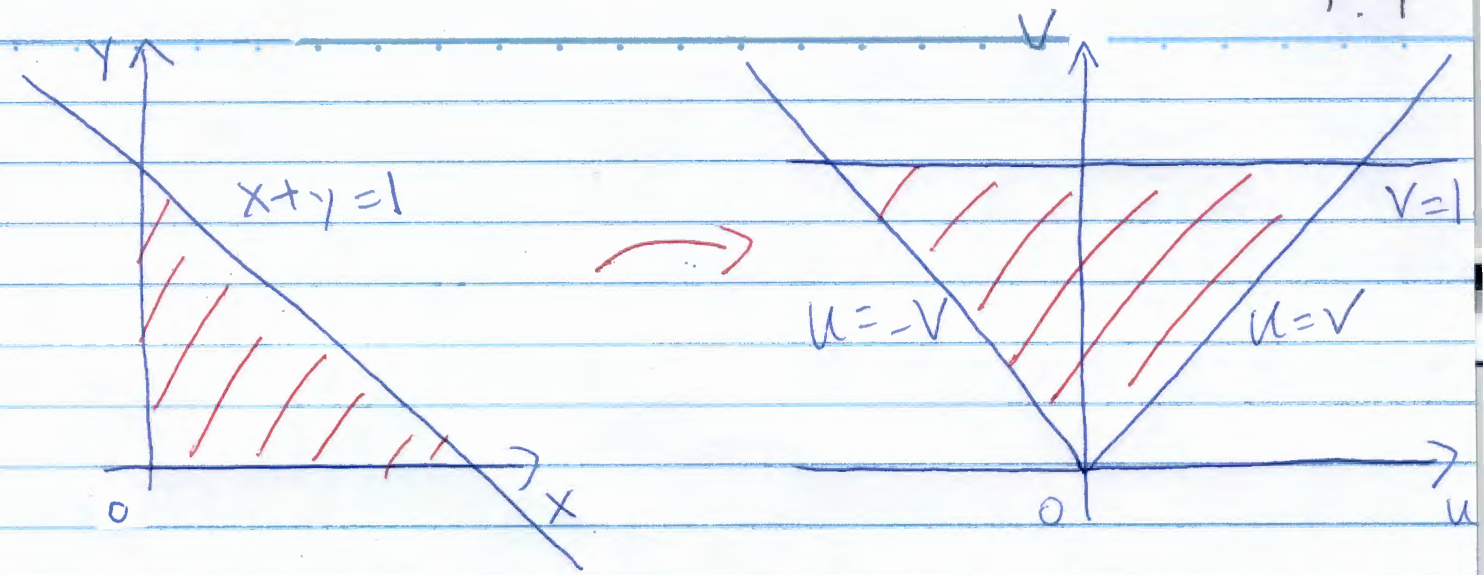
Ans: Use the transformation rotation + enlargement

$$\begin{cases} u = x-y \\ v = x+y \end{cases} \quad \left(\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$x+y=1 \longleftrightarrow v=1$

$x=0,\ y\in[0,1] \longleftrightarrow u=-y=-v,\ y\in[0,1]$

$y=0,\ x\in[0,1] \longleftrightarrow u=x=v,\ x\in[0,1]$



A in xy-plane   A in uv-plane

Note that $\det \dfrac{\partial(x,y)}{\partial(u,v)} = \left[\det \dfrac{\partial(u,v)}{\partial(x,y)}\right]^{-1}$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}^{-1}$$

$$= \frac{1}{2}$$

$$\therefore \iint_A e^{\frac{x-y}{x+y}} dx\,dy = \int_0^1 \int_{-v}^{v} e^{\frac{u}{v}} \left(\frac{1}{2}\right) du\,dv$$

$$= \int_0^1 \frac{v}{2} \int_{-v}^{v} e^{\frac{u}{v}} d\left(\frac{u}{v}\right) dv$$

$$= \int_0^1 \frac{v}{2} \left(e^1 - e^{-1}\right) dv$$

$$= \frac{1}{4}\left(e^1 - e^{-1}\right)$$

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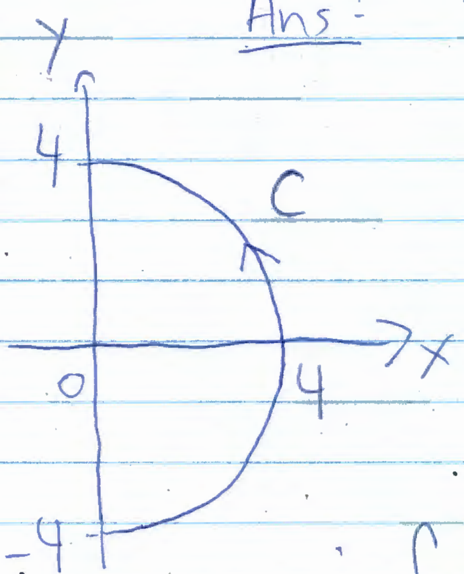
Line Integral:

Defn: Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a scalar function. Then the line integral of f along a curve C is defined by

$$\int_C f\,ds = \int_a^b f(s(t))\,\|s'(t)\|\,dt,$$

where $s:[a,b]\to\mathbb{R}^2$ is a parametrization of the curve C.

Example: 1) $\int_C xy\,ds$, where C is the right half of the circle $x^2+y^2=16$ in anticlockwise direction.



Ans: Parametrize C by $s:[-\frac{\pi}{2},\frac{\pi}{2}]\to\mathbb{R}^2$,

$$s(\theta) = (4\cos\theta, 4\sin\theta).$$

$$\Rightarrow s'(\theta) = (-4\sin\theta, 4\cos\theta)$$

$$\|s'(\theta)\| = \sqrt{(-4\sin\theta)^2 + (4\cos\theta)^2} = 4$$

$$\therefore \int_C xy\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos\theta)(4\sin\theta)(4)\,d\theta$$

$$= 64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,d\sin\theta$$

$$= 64\left[\frac{\sin^2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 0$$

Defn: Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field. Then the line integral of F along a curve C is defined by

$$\int_C F \, d\gamma = \int_a^b F(\gamma(t)) \cdot \gamma'(t) \, dt,$$

where $\gamma: [a,b] \to \mathbb{R}^2$ is a parametrization of the curve C.

Prop: For any smooth curve C parametrized by $\gamma: [0,1] \to \mathbb{R}^2$, if $F = \nabla f$ for some $f: \mathbb{R}^2 \to \mathbb{R}$,

$$\int_C F \, d\gamma = \int_C \nabla f \, d\gamma = f(\gamma(1)) - f(\gamma(0))$$

i.e. F is conservative.

Pf: Let $\gamma(t) = (x(t), y(t))$, $t \in [0,1]$.

Then
$$\begin{aligned}
\int_C F \, d\gamma &= \int_0^1 (f_x, f_y)(x', y') \, dt \\
&= \int_0^1 \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) dt \\
&= \int_0^1 \frac{d}{dt} f(\gamma(t)) \, dt \\
&= f(\gamma(1)) - f(\gamma(0)) \quad \blacksquare
\end{aligned}$$

Defn: (Potential Function and Gradient Field)

- Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field. If $\exists$ a C^2 function $f: \mathbb{R}^2 \to \mathbb{R}$ s.t. $F = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$, then F is called a gradient field and f is called a potential function of F.